Variations on a result of Erdös and Surányi

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1 Erdös-Surányi sequences

In this section we present a special class of sequences of distinct positive integers, which give special representations of the integers. We say that a sequence of distinct positive integers $\{a_m\}_{m\geq 1}$ is a *Erdös-Surányi sequence* if every integer may be written in the form

$$\pm a_1 \pm a_2 \pm \cdots \pm a_n$$

for some choices of signs $+$ and $-$, in infinitely many ways.

As a general example of Erdös-Surányi sequences we mention, for every $k \in \mathbb{N}$, $a_n = n^k$ (see J.Mitek,'79,[16]). For instance, for $k = 1$ we have the representation

$$m = (-1+2)+(-3+4)+\cdots+(-(2m-1)+2m)+\cdots+ \underbrace{[(n+1)-(n+2)-(n+3)+(n+4)]}_{=0}$$

For $k = 2$, we have the original result of Erdös and Surányi mentioned as a problem in the book [12] and as Problem 250 in the book of W.Sierpnski [18]. A standard proof is based on the identity $4 = (m+1)^2 - (m+2)^2 - (m+3)^2 + (m+4)^2$ and on the basis cases

$$0 = 1^2 + 2^2 - 3^2 + 4^2 - 5^2 - 6^2 + 7^2, \; 1 = 1^2$$
$$2 = -1^2 - 2^2 - 3^2 + 4^2, \; 3 = -1^2 + 2^2.$$

For $k = 3$, one may use the identity

$$-(m+1)^3 + (m+2)^3 + (m+3)^3 - (m+4)^3$$
$$+(m+5)^3 - (m+6)^3 - (m+7)^3 + (m+8)^3 = 48$$

and induction with a basis step for the first 48 positive integers.

An interesting example of Erdös-Surányi sequence is given by the squares of the odd integers. The proof by step 16 induction is based on the identity
$16 = (2m+5)^2 - (2m+3)^2 - (2m+1)^2 + (2m-1)^2$, and on the basis cases

$0 = -1^2 + 3^2 + 5^2 - 7^2 + 9^2 - 11^2 - 13^2 + 15^2$
$1 = 1^2$
$2 = 1^2 + 3^2 + 5^2 - 7^2 + 9^2 - 11^2 - 13^2 + 15^2$
$3 = 1^2 - 3^2 + 5^2 + 7^2 + 9^2 + 11^2 + 13^2 - 15^2 - 17^2 - 19^2 + 21^2$
$4 = -1^2 - 3^2 - 5^2 - 7^2 + 9^2 - 11^2 - 13^2 + 15^2 - 17^2 + 19^2$
$5 = 1^2 + 3^2 + 5^2 + 7^2 + 9^2 + 11^2 + 13^2 + 15^2 - 17^2 - 19^2 - 21^2$
$-23^2 - 25^2 + 27^2 + 29^2$
$6 = -1^2 - 3^2 + 5^2 - 7^2 - 9^2 + 11^2$
$7 = 1^2 + 3^2 + 5^2 + 7^2 + 9^2 + 11^2 + 13^2 + 15^2 + 17^2 - 19^2 + 21^2$
$-23^2 - 25^2 - 27^2 + 29^2$
$8 = -1^2 + 3^2$

$9 = -1^2 - 3^2 + 5^2 - 7^2 - 9^2 - 11^2 - 13^2 - 15^2 - 17^2 + 19^2 -$
$-21^2 - 23^2 - 25^2 - 27^2 + 29^2 + 31^2 + 33^2$
$10 = 1^2 + 3^2$
$11 = -1^2 - 3^2 + 5^2 - 7^2 - 9^2 - 11^2 - 13^2 - 15^2 + 17^2 - 19^2 -$
$-21^2 + 23^2 + 25^2$
$12 = -1^2 - 3^2 - 5^2 - 7^2 + 9^2 + 11^2 - 13^2 + 15^2 + 17^2$
$13 = -1^2 - 3^2 - 5^2 - 7^2 + 9^2 + 11^2 - 13^2 - 15^2 + 17^2$
$14 = -1^2 - 3^2 - 5^2 + 7^2$
$15 = -1^2 - 3^2 + 5^2$

The above examples of Erdös-Surányi sequences are particular cases of the following result

Theorem 1.(M.O. Drimbe, '88, [11]) *Let $f \in \mathbb{Q}[X]$ be a polynomial such that for any $n \in \mathbb{Z}$, $f(n)$ is an integer. If the greatest common factor of the terms of the sequence $\{f(n)\}_{n\geq 1}$ is equal to* 1, *then $\{f(n)\}_{n\geq 1}$ is an Erdös-Surányi sequence.*

Using this result, we can obtain other examples of Erdös-Surányi sequences : $a_n = (an-1)^k$ for any $a \geq 2$, and $a_n = \binom{n+s}{s}$ for any $s \geq 2$. Note that it is difficult to obtain a proof by induction for these sequences, similar to those given for the previous examples.

2 Some general results

Recall that a sequence of distinct positive integers is *complete* if every positive integer can be written as a sum of some distinct of its terms. An important result concerning the Erdös-Surányi sequences is the following

Theorem 2.(M.O. Drimbe, '83, [10]) *Let* $\{a_m\}_{m\geq 1}$ *be a sequence of distinct positive integers such that* $a_1 = 1$ *and for every* $n \geq 1$, $a_{n+1} \leq a_1 + \cdots + a_n + 1$. *If the sequence contains infinitely many odd integers, then it is a Erdös-Surányi sequence.*

Unfortunately, the sequences mentioned in Section 1 do not satisfy the condition $a_{n+1} \leq a_1 + \cdots + a_n + 1, n \geq 1$, in Theorem 2 so, we cannot use this result to prove they are Erdös-Surányi sequences.

The following integral formula shows the number of representations of an integer for a fixed n.

Theorem 3. (D. Andrica and D. Văcărețu,'06, [4]) *Given a Erdös-Surányi sequence $\{a_m\}_{m\geq 1}$, then the number of representations of $k \in [-u_n, u_n]$, where $u_n = a_1 + \cdots + a_n$, in the form $\pm a_1 \pm a_2 \pm \cdots \pm a_n$, for some choices of signs $+$ and $-$, denoted here by $A_n(k)$, is given by*

$$A_n(k) = \frac{2^n}{\pi} \int_0^{\pi} \cos(kt) \prod_{j=1}^{k} \cos(a_j t) dt. \tag{1}$$

3 The signum equation

For an Erdös-Surányi sequence $\mathbf{a} = \{a_m\}_{m\geq 1}$, the *signum equation* of $\mathbf{a}$ is

$$\pm a_1 \pm a_2 \pm \cdots \pm a_n = 0. \tag{2}$$

For a fixed integer n, a *solution* to the signum equation is a choice of signs $+$ and $-$ such that (2) holds.
Denote by $S_{\mathbf{a}}(n)$ the number of solutions of the equation (2).
Clearly, if 2 does not divide u_n, where $u_n = a_1 + \cdots + u_n$, then we have $S_{\mathbf{a}}(n) = 0$.
Here are few equivalent properties for $S_{\mathbf{a}}(n)$ (see [3]).

1. $S_{\mathbf{a}}(n)/2^n$ is the unique real number α having the property that the function $f : \mathbb{R} \to \mathbb{R}$, defined by

$$f(x) = \begin{cases} \cos\frac{a_1}{x}\cos\frac{a_2}{x}\cdots\cos\frac{a_n}{x} & \text{if } x \neq 0 \\ \\ \alpha \ \text{ if } \ x = 0, \end{cases}$$

is a derivative.

2. $S_{\mathbf{a}}(n)$ is the term not depending on z in the development of

$$(z^{a_1} + \frac{1}{z^{a_1}})(z^{a_2} + \frac{1}{z^{a_2}})\cdots(z^{a_n} + \frac{1}{z^{a_n}}).$$

3. $S_{\mathbf{a}}(n)$ is the coefficient of $z^{u_n/2}$ in the polynomial

$$(1 + z^{a_1})(1 + z^{a_2})\cdots(1 + z^{a_n}).$$

4. The following integral formula holds

$$S_{\mathbf{a}}(n) = \frac{2^{n-1}}{\pi} \int_0^{2\pi} \cos a_1 t \cos a_2 t \cdots \cos a_n t dt. \tag{3}$$

5. $S_{\mathbf{a}}(n)$ is the number of ordered bipartitions into classes having equal sums of the set $\{a_1, a_2, \cdots, a_n\}$.

6. $S_{\mathbf{a}}(n)$ is the number of partitions of $u_n/2$ into distinct parts, if 2 divides u_n, and $S_{\mathbf{a}}(n) = 0$ otherwise.

7. $S_{\mathbf{a}}(n)$ is the number of distinct subsets of $\{a_1, a_2, \cdots, a_n\}$ whose elements sum to $u_n/2$ if 2 divides u_n, and $S_{\mathbf{a}}(n) = 0$ otherwise.

To study the asymptotic behavior of $S_{\mathbf{a}}(n)$, when $n \to \infty$, is a very challenged problem. For instance, for the sequence $a_n = n^k, k \geq 2$, in this moment we don't have a proof for the relation

$$\lim_{n\to\infty} \frac{S_k(n)}{2^n n^{-\frac{2k+1}{2}}} = \sqrt{\frac{2(2k+1)}{\pi}},$$

where $S_k(n)$ stands for $S_{\mathbf{a}}(n)$ in this case. For $k = 1$ the previous relation was called the Andrica-Tomescu Conjecture [3], and it was recently proved by B.Sullivan [21].

The n-range of an Erdös-Surányi sequence

For a fixed $n \geq 1$, define the *n-range* $\mathcal{R}_{\mathbf{a}}(n)$ of $\mathbf{a} = \{a_m\}_{m\geq 1}$ to be the set consisting in all integers

$$\pm a_1 \pm a_2 \pm \cdots \pm a_n. \tag{4}$$

Clearly, for every $n \geq 1$, the n-range $\mathcal{R}_{\mathbf{a}}(n)$ is a symmetric set with respect 0.

For instance, to determine the range $\mathcal{R}_1(n)$ for the sequence $a_m = m$, was a 2011 Romanian Olympiad problem. Let us include an answer to this problem. The greatest element of the set $\mathcal{R}_1(n)$ is the triangular number $T_n := 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$, and the smallest element of $\mathcal{R}_1(n)$ is clearly $-T_n$. Also, the difference of any two elements of $\mathcal{R}_1(n)$ is an even number. Hence all elements of $\mathcal{R}_1(n)$ are of the same parity.
We claim that

$$\mathcal{R}_1(n) = \{-T_n, -T_n + 2, \cdots, T_n - 2, T_n\}. \tag{5}$$

Let us define a map on the elements of $\mathcal{R}_1(n) \setminus \{T_n\}$ having values in $\mathcal{R}_1(n)$. First, if $x \in \mathcal{R}_1(n) \setminus \{T_n\}$ is an element for which the writing begins with -1, then by changing -1 by $+1$, we get $x + 2 \in \mathcal{R}_1(n)$. If the writing of x begins with $+1$, then consider the first term in the sum with sign $-$. Such a term exists unless $x = T_n$. In this case we have

$$x = 1 + 2 + \cdots + (j-1) - j \pm \cdots \pm n.$$

By changing the signs of terms $j-1$ and j, it follows that $x + 2 \in \mathcal{R}_1(n)$. This shows the claim in (5).

For $k = 2$ the situation with the n-range $\mathcal{R}_2(n)$ is almost the same as in case $k = 1$ but there is an interesting new phenomenon, although expected since $\{m^2\}_{m\geq 1}$ is a Erdös-Surányi sequence as we have seen. Let us define the set

$$R_2(n) = \{-\sum\nolimits_2(n), -\sum\nolimits_2(n) + 2, \cdots, \sum\nolimits_2(n) - 2, \sum\nolimits_2(n)\},$$

where $\sum_2(n) = 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$.

Theorem 3.(D. Andrica and E.J. Ionașcu,'13, [2]) *For* $n \in \mathbb{N}$, $\mathcal{R}_2(n) = R_2(n) \setminus \mathcal{E}_2(n)$, *where*

$$\mathcal{E}_2(n) = \{\pm(\sum\nolimits_2(n) - 2j) : j \in E\} \text{ and}$$

$$E := \{2, 3, 6, 7, 8, 11, 12, 15, 18, 19, 22, 23, 24, 27, 28, 31, 32, 33,$$

$$43, 44, 47, 48, 60, 67, 72, 76, 92, 96, 108, 112, 128\}. \tag{6}$$

For a sequence of distinct positive integers $\mathbf{a} = \{a_m\}_{m\geq 1}$ define the *exceptional set* of $\mathbf{a}$ to be the set $E(\mathbf{a})$ consisting in all positive integers that cannot be represented as a sum of distinct terms of $\mathbf{a}$.

The exceptional set of the sequence $a_m = m^2$ is

$$E := \{2, 3, 6, 7, 8, 11, 12, 15, 18, 19, 22, 23, 24, 27, 28, 31, 32, 33,$$
$$43, 44, 47, 48, 60, 67, 72, 76, 92, 96, 108, 112, 128\}. \tag{7}$$

For $k = 3$, a similar result can be stated. The list of the numbers (A001476) which cannot be represented as a sum of distinct cubes has 2788 terms. This was obtained by R. E. Dressler and T. Parker in [9].

For $k = 4$ the exceptional set of numbers (A046039) has 889576 elements.

In [20], it is denoted by $P_n(k)$ the number of partitions of k into distinct parts from $1, 2^n, 3^n, \ldots$, and it is proved that for each n there are only a finite number of integers which are not the sums of distinct n^{th} powers. That is, there is a positive integer N_n depending only on n such that $P_n(k) > 0$ for all $k > N_n$. This result was extended by H. E. Richert ([17]) to a more general class of sequences.

As we have seen for the sequence of squares and of cubes, it is challenging to determine the exceptional set for a given Erdös-Surányi sequence. For a better understanding of the difficulty of this problem, we mention here few more examples.

Example 1 : Triangular numbers

The sequence $\mathbf{t}$ of triangular numbers $T_n = \frac{n(n+1)}{2}$, $(n \geq 1)$, satisfies for every $m \geq 1$ the relation $T_{m+3} - T_{m+3} - T_{m+3} + T_m = 2$. Since we may write $1 = T_1$ and $2 = -T_1 + T_2$, it follows by induction that $\mathbf{t}$ is an Erdös-Surányi sequence. According to the result of H. E. Richert ([17]), its exceptional set is $E(\mathbf{t}) = \{2, 5, 8, 12, 23, 33\}$.

Example 2 : The sequence of primes

It is also known that the sequence of primes is an Erdös-Surányi sequence. A nice proof based on Theorem 1 combined with Bertrand's postulate is given by M.O. Drimbe ([10], Proposition 4). According to the result of R. E. Dressler [8], every positive integer, except $1, 2$ and 6, can be written as the sum of distinct primes, that is the exceptional set of the sequence **p** of primes is $E(\mathbf{p}) = \{1, 4, 6\}$.

Example 3 : Fibonacci numbers

Finally, it is not difficult to check that the hypotheses in Theorem 1 are satisfied for the Fibonacci sequence $F_0 = 0$, $F_1 = 1$, $F_{n+2} = F_{n+1} + F_n$, $n \geq 0$. Thus, $\{F_n\}$ is an Erdös-Surányi sequence. On the other hand, it is more or less known Zeckendorf's theorem in [24], which states that every positive integer can be represented uniquely as the sum of one or more distinct Fibonacci numbers in such a way that the sum does not include two consecutive Fibonacci numbers. Such a sum is called *Zeckendorf representation* and it is related to the Fibonacci coding of a positive integer. In this case, the exceptional set $E(\mathbf{f})$ of the Fibonacci sequence, say $\mathbf{f}$, is the empty set.

Example 4 : Odd squares

We have seen in the first section that an interesting example of Erdös-Surányi sequence is given by $a_m = (2m-1)^2$, the squares of the odd integers. The exceptional set seems to contain 534 numbers (OEIS, the sequence A167703), but in this moment we don't know a proof for this property

$$2, 3, 4, 5, 6, 7, 8, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24,$$

$$27, 28, 29, 30, 31, 32, 33, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48,$$

$$51, 52, 53, 54, 55, 56, 57, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71,$$

$$72, 73, 76, 77, 78, 79, 80, 85, \cdots, 1922$$

Conjecture 1. *Every integer* ≥ 1923 *can be written as a sum of distinct odd squares.*

For instance we can write
$1923 = 11^2 + 29^2 + 31^2, 1924 = 1^2 + 11^2 + 29^2 + 31^2, 1925 = 1^2 + 3^2 + 15^2 + 27^2 + 31^2, 1926 = 1^2 + 3^2 + 11^2 + 15^2 + 27^2 + 29^2$

Conjecture 2. *The exceptional set of every Erdös-Surányi sequence is finite.*

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